Effects of Induced Magnetic Fields in a Typical Railgun Problem

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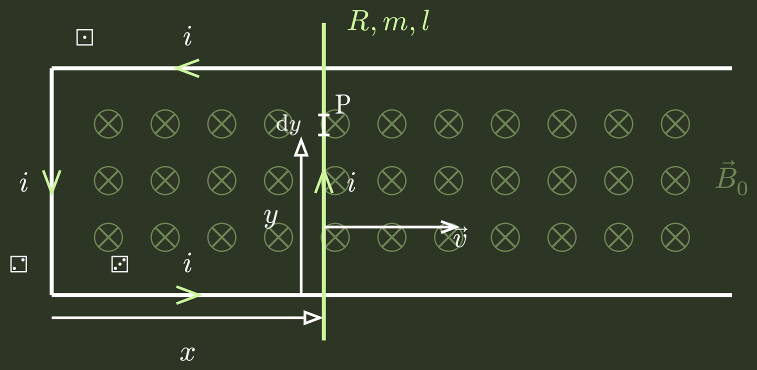

In these types of problems, also solved in my Questions page, we do not consider the fact that the conducting wires used here, i.e., ⚀, ⚁, ⚂, carry current, and hence would create their own magnetic fields. Here, we have a look at what would change if we **do** consider these magnetic fields, and whether or not they are truly negligible.

Assumptions: The conducting wires used have negligible resistance. Only the rod(highlighted) has a considerable resistance of R. The width of the setup, which is also the length of the rod, is l. The mass of the rod considered is m. It is initially pushed to the right with an initial velocity of v_0, which at a general moment, is v. Let the distance of the rod from the leftmost wire(Wire ⚁) be x. Consider a general point P, at some distance y from the lower wire(Wire ⚂). A current i will be generated due to electromagnetic induction. A constant magnetic field of magnitude B_0 is present with the direction being into the plane of the page.

By using Biot-Savart's law, we can get the magnetic fields due to wires ⚀, ⚁, ⚂ at point P

$$B_{⚀} = \frac{\mu_0 i}{4\pi(l-y)}\left[\frac{x}{\sqrt{x^2+(l-y)^2}}\right]\odot \tag{1a}$$

$$B_{⚁} = \frac{\mu_0 i}{4\pi x}\left[\frac{y}{\sqrt{x^2+y^2}}+\frac{l-y}{\sqrt{x^2+(l-y)^2}}\right]\odot \tag{1b}$$

$$B_{⚂} = \frac{\mu_0 i}{4\pi y}\left[\frac{x}{\sqrt{x^2+y^2}}\right]\odot \tag{1c}$$

Where the $\odot$ represents that they all have the directions of the magnetic field out of the plane. Hence, the total field at any distance x, height y, at a point P is:

$$\mathcal{B} = B_0 - [B_{\boxdot} + B_{\boxdot} + B_{\boxdot}] \tag{2}$$

Which can be simplified to,

$$\vec{\mathcal{B}} = B_0 - \frac{\mu_0 i}{4\pi}\left[\sqrt{\frac{1}{x^2}+\frac{1}{y^2}} + \sqrt{\frac{1}{x^2}+\frac{1}{(l-y)^2}}\right] \otimes \tag{3}$$

With the direction of the vector pointing in to the plane.

Now, we can find the induced electromagnetic force:

$$\mathcal{E} = \int_r^{l-r} \vec{v} \times \vec{\mathcal{B}} \cdot \mathrm{d}\vec{y} \tag{4}$$

Note our limits of integration. They go from r(the radius of each conducting wire) to $l-r$. This is because the magnetic field of a line current tends to infinity as the distance from the wire goes to zero. Hence, we consider some small, but non-zero radius of the wire, r.

Since $\vec{v}$, $\vec{\mathcal{B}}$ and $\mathrm{d}\vec{y}$ are mutually perpendicular, the triple scalar product is simply the products of their magnitudes. Further, the velocity $\vec{v}$ is independent of height y. Therefore, it can be taken out of the integral. So,

$$\mathcal{E} = v\int_r^{l-r} \mathcal{B}\,\mathrm{d}y \tag{5a}$$

$$= vB_0 l - \frac{\mu_0 i}{4\pi}\int_r^{l-r}\left[\sqrt{\frac{1}{x^2}+\frac{1}{y^2}} + \sqrt{\frac{1}{x^2}+\frac{1}{(l-y)^2}}\right]\mathrm{d}y \tag{5b}$$

$$= vB_0 l - \frac{\mu_0 i}{4\pi}\left[\sqrt{1+\left(\frac{y}{x}\right)^2} - \coth^{-1}\left(\sqrt{1+\left(\frac{y}{x}\right)^2}\right)\right.$$

$$\left. - \sqrt{1+\left(\frac{l-y}{x}\right)^2} + \coth^{-1}\left(\sqrt{1+\left(\frac{l-y}{x}\right)^2}\right)\right]\Bigg|_r^{l-r} \tag{5c}$$

In Equation 5b, we see that i, even though it varies with time, does NOT vary with y, and hence can be taken out of the integral.

Here, for simplicity, we shall define a function:

$$\Lambda_x(a) \stackrel{\text{def}}{=} \sqrt{1+\left(\frac{a}{x}\right)^2} \tag{6}$$

Hence, we get:

$$\mathcal{E} = vB_0 l - \frac{\mu_0 i}{2\pi}[\Lambda_x(l-r) - \Lambda_x(r) + \coth^{-1}(\Lambda_x(r)) - \coth^{-1}(\Lambda_x(l-r))] \tag{7}$$

Further, since this is an only resistance circuit, Ohm's law is valid:

$$\mathcal{E} = iR \tag{8}$$

Substituting this, we can solve for the current i, and we get,

$$i = \frac{vB_0 l}{R + \frac{\mu_0}{2\pi}[\Lambda_x(l-r) - \Lambda_x(r) + \coth^{-1}(\Lambda_x(r)) - \coth^{-1}(\Lambda_x(l-r))]} \tag{9}$$

Compare this with what we would have gotten if we neglected the magnetic fields produced by the wires.

$$i = vB_0 l \tag{10}$$

In Equation 9, we see that for the whole term in the bracket, there is a common factor of $\frac{\mu_0}{2\pi}$, which has a value of exactly[1] $2 \cdot 10^{-7}\mathrm{NA}^{-2}$, which is a *tiny* quantity. Hence that term basically makes no difference to the net force on the rod.

For a better view, here is the graph of both functions(Equation 9, and Equation 10):

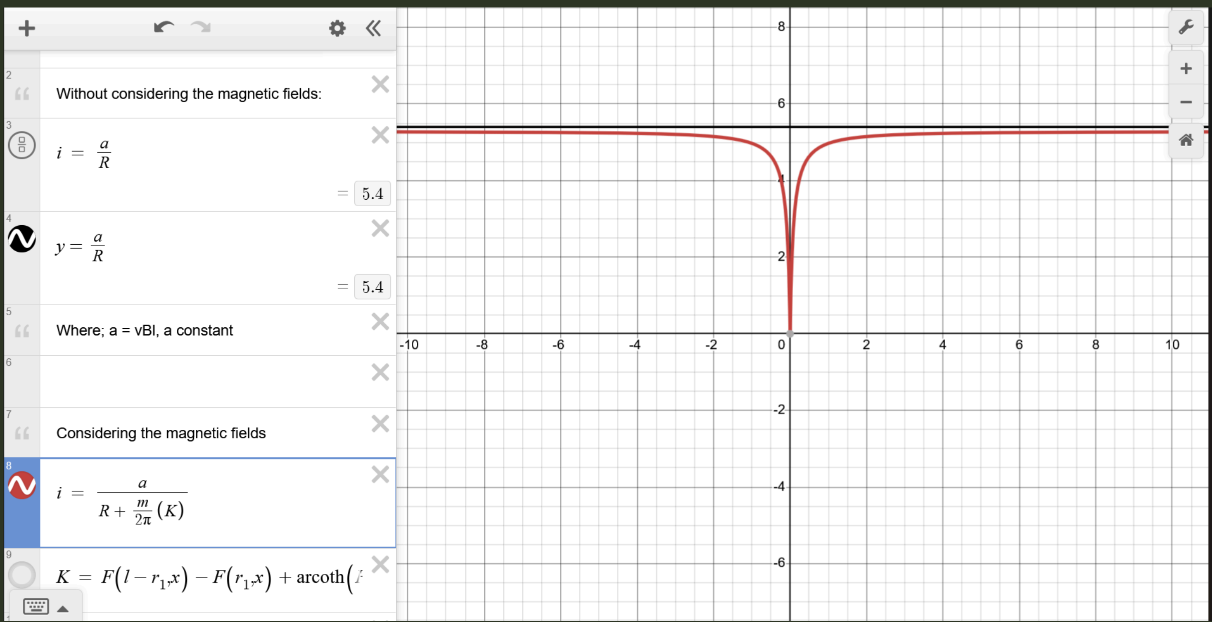

Figure 2: Interactive Plot

Here, for clarity, I have made the value of $\frac{\mu_0}{2\pi}$ quite large(≈ 0.01), so that one can see a clear distinction between the red(Equation 9) and the black(Equation 10) curves. In the Interactive Plot, you can change a couple of values to see the difference. Try it out!

In the graph, I have used different variable names for ease of typing the equation in the plotter. Note that here the X axis represents the distance x from our discussions, and the Y axis represents current i in the circuit at a given moment of time. This is only for us to note the nature of the graph, not the exact values.

Conclusion

From the discussion above, we can safely say that the effect of the wires' magnetic fields are, in fact, *very* negligible. This is mainly due to the fact that μ_0 is such a small quantity. Even then, in our expression, the coefficient of $\mu_0/2\pi$ is also small. $\Lambda_x(r)$ is incredibly small, given that r is small, which was one of our considerations. $\Lambda_x(l-r)$ is also small, since in both the equations, x is in the denominator. So, with an increase in x, we simply see an overall decrease the that term. So, considering i to be almost the same, the other equations derived in similar problems will also be *almost* same.

[1]According to old standards, but it is approximately still the same.