Equation of Curve of a Hanging Chain, and its Center of Mass

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> Find the equation of a chain, hanging from two points symmetrically, such that the height from the hinge to the bottom-most point of the chain is h. Let the mass be of uniform density, with mass per unit length as λ.
> Find the center of mass of this chain.
> Further, what happens to the position of the center of mass if we were to pull the bottom most point of the chain by some external force such that the shape of the chain essentially becomes triangular?

First, we shall find out the equation of the curve, then find the length of the chain s(since we are only given the height h). Next, we'll find the center of mass of the chain, and finally the center of mass when it is triangular.

Equation of the Curve

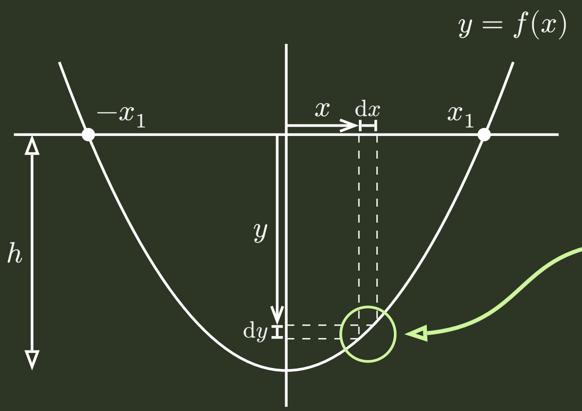


Figure 1: Sketch of the curve

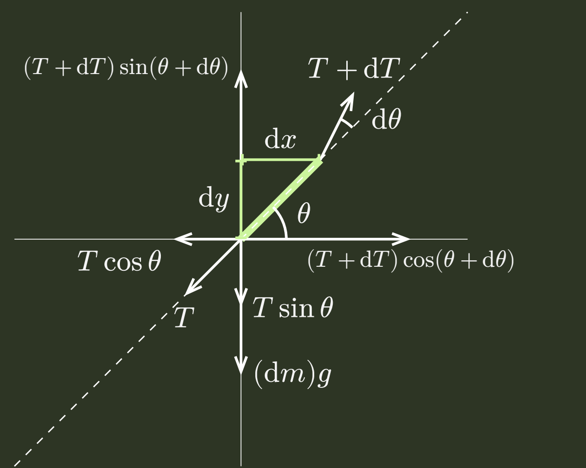


Figure 2: Free body diagram(FBD) of the highlighted $\mathrm{d}m$ piece

As we have all seen hanging cables/chains, we can safely assume that $f(x)$ is symmetric about the Y axis.

We shall now balance forces along the X axis(as seen in the FBD):

$$T\cos\theta = (T + \mathrm{d}T)\cos(\theta + \mathrm{d}\theta) \tag{1a}$$

$$= (T + \mathrm{d}T)[(\cos\theta)(\cancelto{1}{\cos\mathrm{d}\theta}) - (\sin\theta)(\cancelto{\mathrm{d}\theta}{\sin\mathrm{d}\theta})] \tag{1b}$$

$$\cancel{T\cos\theta} = \cancel{T\cos\theta} - T\sin\theta\,\mathrm{d}\theta + \mathrm{d}T\cos\theta + \cancelto{0}{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}T} \tag{1c}$$

$$T\sin\theta\,\mathrm{d}\theta = \mathrm{d}T\cos\theta \tag{1d}$$

$$\int_0^\theta \tan\theta \,\mathrm{d}\theta = \int_{T_0}^{T} \frac{\mathrm{d}T}{T} \tag{1e}$$

$$\Rightarrow T = T_0 \sec\theta \tag{1f}$$

Here, we assume that when $\theta = 0$, which is the bottom most point, the tension is some constant T_0.

Note that, $\mathrm{d}m = \lambda \,\mathrm{d}s$, where $\mathrm{d}s$ is the length of the small mass considered. This can be re-written using Pythagoras' theorem as $\mathrm{d}m = \lambda \,\mathrm{d}x \sqrt{1 + (\mathrm{d}y/\,\mathrm{d}x)^2}$

We shall now balance forces along the Y axis:

$$T \sin\theta + g \,\mathrm{d}m = (T + \mathrm{d}T)\sin(\theta + \mathrm{d}\theta) \tag{2a}$$

$$\cancel{T\sin\theta} + \lambda g \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \cancel{T\sin\theta} + T\cos\theta\,\mathrm{d}\theta + \sin\theta\,\mathrm{d}T + \cancelto{0}{\cos\theta\,\mathrm{d}\theta\,\mathrm{d}T} \tag{2b}$$

Using the fact that $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta$(Which can be seen from the Free Body Diagram), and the following:

$$T = T_0 \sec\theta \tag{3a}$$

$$\Rightarrow \mathrm{d}T = T_0 \sec\theta \tan\theta \,\mathrm{d}\theta \tag{3b}$$

We get,

$$\lambda g \sec\theta \,\mathrm{d}x = T_0 \,\mathrm{d}\theta + T_0 \tan^2\theta \,\mathrm{d}\theta \tag{4a}$$

$$\lambda g \cancel{\sec\theta} \,\mathrm{d}x = T_0 \sec^{\cancel{2}}\theta \,\mathrm{d}\theta \tag{4b}$$

$$\lambda g \int_0^x \mathrm{d}x = T_0 \int_0^\theta \sec\theta \,\mathrm{d}\theta \tag{4c}$$

$$\lambda g x = T_0 \ln|\sec\theta + \tan\theta\,| \tag{4d}$$

$$e^{\frac{\lambda g x}{T_0}} = \sec\theta + \tan\theta \tag{4e}$$

For ease of further calculations, we define a constant,

$$\alpha = \frac{\lambda g}{T_0} \tag{5}$$

Also, from the tiny right-angled triangle in the FBD, we get:

$$\tan\theta = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{6a}$$

$$\sec\theta = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \tag{6b}$$

Using these in Equation 4e,

$$e^{\alpha x} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} + \frac{\mathrm{d}y}{\mathrm{d}x} \tag{7a}$$

$$(e^{\alpha x} - y')^2 = 1 + (y')^2 \tag{7b}$$

$$e^{2\alpha x} + \cancel{(y')^2} - 2e^{\alpha x} y' = 1 + \cancel{(y')^2} \tag{7c}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(e^{\alpha x} - e^{-\alpha x}) = \sinh(\alpha x) \tag{7d}$$

$$\int_0^x \sinh(\alpha x)\,\mathrm{d}x = \int_h^y \mathrm{d}y \tag{7e}$$

Hence, the equation of the curve is,

$$\boxed{\alpha(y+h)+1 = \cosh(\alpha x)} \tag{8}$$

We can also find its roots(x_1 and $-x_1$) by setting $y = 0$,

$$x_1 = \frac{1}{\alpha}\cosh^{-1}(\alpha h + 1) \tag{9}$$

Length of the chain(arc length)

From the tiny triangle in the FBD, we have that:

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} \tag{10a}$$

$$\mathrm{d}s = \mathrm{d}x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \tag{10b}$$

$$\int \mathrm{d}s = \int_{-x_1}^{x_1} \sqrt{1+\sinh^2(\alpha x)}\,\mathrm{d}x \tag{10c}$$

$$= \int_{-x_1}^{x_1} \cosh(\alpha x)\,\mathrm{d}x \tag{10d}$$

$$s = \frac{2}{\alpha}\sinh(\alpha x_1) \tag{10e}$$

$$= \frac{2}{\alpha}\sinh\left(\cancel{\alpha}\left(\frac{\cosh^{-1}(\alpha h+1)}{\cancel{\alpha}}\right)\right) \tag{10f}$$

$$s = \frac{2}{\alpha}\sqrt{(\alpha h+1)^2 - 1} \tag{10g}$$

Here, we make use of some hyperbolic trigonometric identities:

$$\cosh^2(x) - \sinh^2(x) = 1 \tag{11a}$$

$$\sinh(\cosh^{-1}(x)) = \sqrt{x^2-1} \tag{11b}$$

Center of Mass of Chain

By symmetry, the x_{CM} of the chain will be zero. y_{CM} is given by:

$$y_{\mathrm{CM}} = \frac{\int y\,\mathrm{d}m}{\int \mathrm{d}m} \tag{12a}$$

$$= \frac{\int y\cancel{\lambda}\,\mathrm{d}x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}}{\int \cancel{\lambda}\,\mathrm{d}x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}} \tag{12b}$$

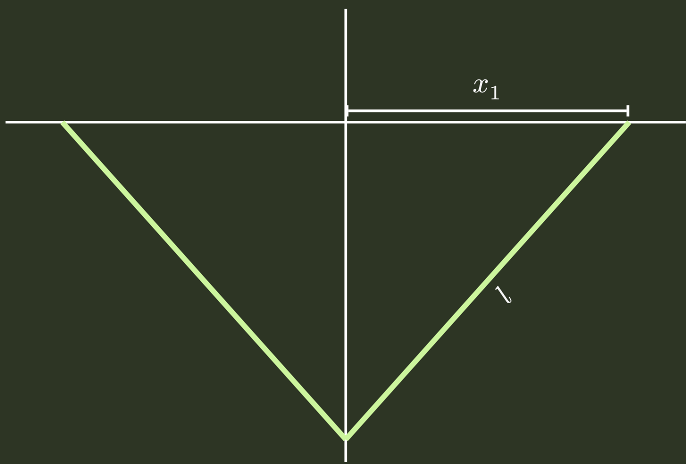
$$= \frac{\int_{-x_1}^{x_1}\left(\frac{\cosh(\alpha x)-1}{\alpha} - h\right)\cosh(\alpha x)\,\mathrm{d}x}{\int_{-x_1}^{x_1}\cosh(\alpha x)\,\mathrm{d}x} \tag{12c}$$

$$= \frac{\left[\frac{(\cosh(\alpha x)-2\alpha h-2)\sinh(\alpha x)+\alpha x}{2\alpha^2}\right]_{-x_1}^{x_1}}{s} \tag{12d}$$

$$= \frac{1}{2\alpha}\left(\frac{\cosh^{-1}(\alpha h+1)}{\sqrt{(\alpha h+1)^2-1}} - (\alpha h+1)\right) \tag{12e}$$

In the last step, we substituted values for x_1 and s which we got earlier.

Center of Mass when Triangular



For simplicity, we define:

$$l \overset{\text{def}}{=} \frac{1}{\alpha}\sqrt{(\alpha h+1)^2-1} = \frac{s}{2} \tag{13}$$

The equations of the lines can be written as,

$$y = \sqrt{l^2 - x_1^2}\left(\frac{|x|}{x_1} - 1\right) \tag{14a}$$

$$\text{or,} \quad y = m\left(\frac{|x|}{x_1} - 1\right) \tag{14b}$$

$$\Rightarrow y' = \pm\frac{m}{x_1} \tag{14c}$$

$$\Leftrightarrow (y')^2 = \left(\frac{m}{x_1}\right)^2 \tag{14d}$$

Now to find the center of mass,

x_{CM} will be zero by symmetry again, for y_{CM},

$$y_{\text{CM}} = \frac{\int y\,\mathrm{d}m}{\int \mathrm{d}m} \tag{15a}$$

$$= \frac{\int y\cancel{\lambda}\,\mathrm{d}x\sqrt{1+(y')^2}}{\int \cancel{\lambda}\,\mathrm{d}x\sqrt{1+(y')^2}} \tag{15b}$$

$$= \frac{m\left(\int_{-x_1}^{x_1} \frac{|x|}{x_1} - 1\,\mathrm{d}x\right)}{\int_{-x_1}^{x_1}\mathrm{d}x} \tag{15c}$$

$$= -\frac{m}{2} \tag{15d}$$

$$= \frac{-\sqrt{l^2 - x_1^2}}{2} \tag{15e}$$

Graph

Plotting the above in Desmos, we see that $y_{(\text{CM - chain})} < y_{(\text{CM - triangle})} \forall \alpha, h \in \mathbb{R}^+$.

The link of the graph. The red curve is the hanging chain, whereas the green one is the triangular one.

Finding T_0

We have defined α as $\frac{\lambda g}{T_0}$, without knowing what T_0 is.

Currently, our equation for the curve depends on the height of the chain from the origin h, and α. But this is not what you will generally know why you just tie some rope. Hence, we can re-write our equations in terms of the length of the chain s, and the hinge positions x_1.

Doing so, we get this trancendental equation,

$$\sinh(\alpha x_1) = \alpha \frac{s}{2} \tag{16}$$

This cannot be solved exactly for α in terms of elementary functions. The best we can do is a Taylor series expansion. Considering 3 terms for $\sinh(\alpha x_1)$, we get,

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \tag{17a}$$

$$\alpha x_1 + \frac{(\alpha x_1)^3}{6} + \frac{(\alpha x_1)^5}{120} = \alpha \frac{s}{2} \tag{17b}$$

$$\Rightarrow \alpha^4 \left(\frac{x_1^5}{60} \right) + \alpha^2 \left(\frac{x_1^3}{3} \right) + (2x_1 - s) = 0 \tag{17c}$$

$$\Rightarrow \alpha = \sqrt{\frac{-\frac{x_1^3}{3} \pm \sqrt{\frac{x_1^6}{9} - 4(2x_1 - s)\left(\frac{x_1^5}{60}\right)}}{x_1^5/30}} \tag{17d}$$

$$T_0 = \frac{\lambda g}{\alpha} \tag{17e}$$