Self Inductance of a Straight Wire

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> Find the self inductance of a straight wire of length l and radius R. Assume the current to be spread equally in the interior of the wire, i.e. current density($\vec{j}$) is constant.

To find this, we shall first split the inductances into an *internal* inducatance($\mathcal{L}_1$) and an *external* inductance($\mathcal{L}_2$).

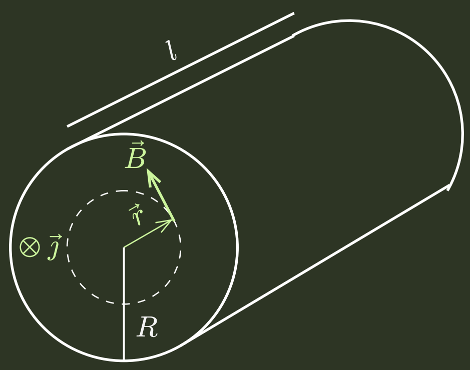
To find *internal* inductance($\mathcal{L}_1$):
This would be the inductance formed within the wire itself. Note that the field inside the wire has a different expression to that outside the wire. To solve for $\mathcal{L}_1$, we shall equate the energies.

Magnetic energy in a region,

$$U = \frac{1}{2\mu_0} \int B^2 \, \mathrm{d}V \tag{1}$$

We shall compare this with the expression for energy in an inductor,

$$U = \frac{1}{2} \mathcal{L} i^2 \tag{2}$$



The magnetic field within a wire carrying current of current density $\vec{j}$,

$$\vec{B} = \frac{\mu_0}{2} (\vec{j} \times \vec{r}) \tag{3}$$

Here, since $\vec{j}$ is constant in magnitude(into the plane of the page), which is always orthogonal to the radius vector $\vec{r}$, the cross product is simply the product of their magnitudes, and since $\vec{x} \times \vec{x} = |\vec{x}|^2$, we get:

$$B^2 = \frac{\mu_0^2 i^2 r^2}{4\pi^2 R^2} \tag{4}$$

We can now integral over the volume of the cylindrical wire to get the total energy *within* the wire.

$$U = \frac{1}{2\mu_0} \frac{\mu_0^2 i^2}{4\pi^2 R^2} \iiint\limits_{000}^{l2\pi R} r^2 (r\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z) \tag{5}$$

Here, r varies from 0 to R, θ varies from 0 to 2π, and z varies from 0 to l. We have used the infinitesimal cylindrical volume element.

Integrating, we get the simplified expression that is independent of R,

$$U = \underbrace{\frac{\mu_0 i^2 l}{16\pi} = \frac{1}{2}\mathcal{L}_1 i^2}_{\text{compare}} \tag{6}$$

$$\Rightarrow \mathcal{L}_1 = \frac{\mu_0 l}{8\pi} \tag{7}$$

Remark: Notice that the *internal* self inductance here is independent of R. Further, the *internal* inductance per unit length is a constant value:

$$\frac{\mathcal{L}_1}{l} = \frac{\mu_0}{8\pi} \tag{8}$$

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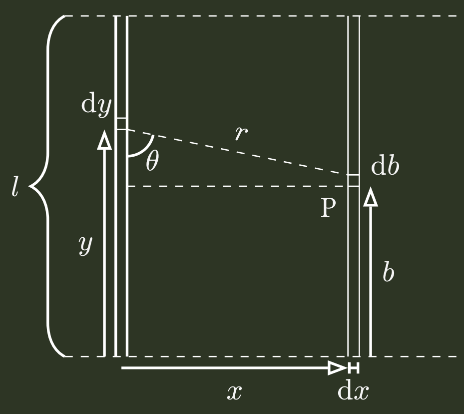
To find *external* inductance($\mathcal{L}_2$):

Here, we shall make use of the fact that the magnetic flux $\Phi = \mathcal{L}_2 i$. We shall calculate the flux as $\Phi = \iint \vec{B} \cdot \mathrm{d}\vec{A}$. Note that $\mathrm{d}A = \mathrm{d}x\, \mathrm{d}b$



We know that the magnetic field of a wire, here at point P is into the plane of the page. Hence, the dot product between the two is simply the product of their magnitudes.

By Biot-Savart's law, the magnetic field at point P, B_P is given by,

$$\mathrm{d}B_\mathrm{P} = \frac{\mu_0}{4\pi} \frac{i\, \mathrm{d}y \sin\theta}{r^2} \tag{9}$$

We can get the total magnetic field at point P by integrating y along the length of the wire.

$$B_\mathrm{P} = \frac{\mu_0 i}{4\pi} \int_0^l \frac{x}{[(y-b)^2 + x^2]^{\frac{3}{2}}} \mathrm{d}y \tag{10}$$

Integrating and substituting limits, we get:

$$B_{\mathrm{P}} = \frac{\mu_0 i}{4\pi} \cancel{x} \left[\frac{l-b}{x^{\cancel{2}} \sqrt{(l-b)^2 + x^2}} + \frac{b}{x^{\cancel{2}} \sqrt{b^2 + x^2}} \right] \tag{11}$$

Now, $\Phi = \iint\limits_{R\,0}^{\infty\,l} B \,\mathrm{d}b \,\mathrm{d}x$. We shall first integrate with respect to b, then x.

$$\Phi = \frac{\mu_0 i}{4\pi} \int_R^\infty \frac{1}{x} \int_0^l \frac{l-b}{\sqrt{(l-b)^2 + x^2}} + \frac{b}{\sqrt{b^2 + x^2}} \,\mathrm{d}b \,\mathrm{d}x \tag{12a}$$

$$= \frac{\mu_0 i}{4\pi} \int_R^\infty \frac{1}{x} 2 \left[\sqrt{x^2 + l^2} - x \right] \mathrm{d}x \tag{12b}$$

$$= \frac{\mu_0 i}{2\pi} \int_R^\infty \frac{\sqrt{x^2 + l^2} - x}{x} \,\mathrm{d}x \tag{12c}$$

$$= \frac{\mu_0 i}{2\pi} \left[\sqrt{x^2 + l^2} - x - l \sinh^{-1}\left(\frac{l}{x} \right) \right] \Bigg|_R^\infty \tag{12d}$$

$$= \frac{\mu_0 i}{2\pi} \left[\cancelto{0}{\sqrt{\infty^2 + l^2} - \infty} - 0 - \sqrt{R^2 + l^2} + R + l \sinh^{-1}\left(\frac{l}{R} \right) \right] \tag{12e}$$

$$\Phi = \frac{\mu_0 i}{2\pi} \left[R + l \sinh^{-1}\left(\frac{l}{R} \right) - \sqrt{R^2 + l^2} \right] \tag{12f}$$

Note that in Equation 12e, the expression evaulating to zero goes to zero in the limit as x tends to infinity. And, $\Phi = \mathcal{L}_2 i$. Comparing,

$$\mathcal{L}_2 = \frac{\mu_0}{2\pi} \left[R\left(1 - \sqrt{1 - \eta^2} \right) + l \sinh^{-1}(\eta) \right] \tag{13}$$

Where $\eta \overset{\text{def}}{=} l/R$

Hence, total self inductance, $\mathcal{L}$,

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \tag{14a}$$

$$\mathcal{L} = \frac{\mu_0}{2\pi} \left[R\left(1 - \sqrt{1 + \eta^2} \right) + l \sinh^{-1}(\eta) + \frac{l}{4} \right] \tag{14b}$$

$$\boxed{\mathcal{L} = \frac{\mu_0 R}{2\pi} \left[\left(1 - \sqrt{1 + \eta^2} \right) + \eta \sinh^{-1}(\eta) + \frac{\eta}{4} \right]} \tag{15}$$

Equation 15 gives us the exact equation for the self inductance of a straight conductor. However, generally the length l of the conductor is much larger than its radius R. So, $\eta \gg 1$. Using this approximation, we can re-write Equation 15 in simpler terms.

First, approximation for $\sinh^{-1} x$, when x is large, i.e. $x \gg 1$

$$\sinh^{-1} x = \ln\left(x + \sqrt{1 + x^2} \right) \tag{16a}$$

$$= \ln\left(x \left(1 + \sqrt{\frac{1}{x^2} + 1} \right) \right) \tag{16b}$$

$$\approx \ln\left(x \left(1 + \sqrt{\cancelto{0}{\frac{1}{x^2}} + 1} \right) \right) \tag{16c}$$

$$\therefore \sinh^{-1} x \approx \ln(2x) \text{ for large } x \tag{16d}$$

Note that in Equation 16c, we make use of the fact that $\eta \gg 1$. Now, Equation 15 can be approximated as(moving from Equation 14b):

$$\mathcal{L} \approx \frac{\mu_0 l}{2\pi}\left[\frac{1}{\eta} - \sqrt{\frac{1}{\eta^2} + 1} + \sinh^{-1}(\eta) + \frac{1}{4}\right] \tag{17a}$$

$$\approx \frac{\mu_0 l}{2\pi}\left[\ln(2\eta) + \frac{1}{\eta} - \frac{3}{4}\right] \tag{17b}$$

$$\boxed{\mathcal{L} \approx \frac{\mu_0 l}{2\pi}\left[\ln(2\eta) - \frac{3}{4}\right]} \tag{17c}$$

Equation 17c is the most commonly found answer for the self inductance of a finite conductor.

Some unanswered questions:

1. Why are the limits of b from 0 to l? As we are calculating flux, shouldn't we integrate over all of space?

References

1. This thread from the Physics Stack Exchange.
2. The Self and Mutual Inductances of Linear Conductors, by Edward B. Rosa. Washington, September 15, 1907. File.
3. Inductance Calculation Techniques – Part II: Approximations. by Marc T. Thompson. Though not exactly a *Calculation Techniques* paper, it possess the approximate equation, Equation 17c, we derived. File.